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PHANTOM SCALAR FIELDS AND ITS PHYSICAL IMPLICATIONS

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Abstract

Equations of state are utilized to explore the possible constrains on dark energy models and their relevant physical quantities such as density parameter, pressure and etc., are investigated. As the situation dictates, physical interpretation and visualizations of the results obtained are performed.

Keywords: phantom scalar field , big rip, anti-big rip, general relativity

Introduction

Cosmological models have developed with complex potentials and have shown that they are rather convenient for the description of the so called phantom cosmology, including such an enigmatic phenomenon. Phantom energy is a hypothetical form of dark energy satisfying the equation of state with $\omega < -1$. It possesses negative kinetic energy, and predicts expansion of the universe in excess of that predicted by a cosmological constant, which leads to a Big Rip. To obtain $\omega < -1$, phantom field with a negative kinetic term may be a simplest implementing and can be regarded as one of interesting possibilities describing dark energy.

All these models of scalar fields lead to the equation of state parameter ω greater than or equal to minus one. It is therefore important to look for theoretical possibilities to describe dark energy with $\omega < -1$ called phantom energy. Phantom thermodynamic leads to a negative entropy (or negative temperature) and the energy density increases to infinity in a finite time, at which point the size of the Universe blows up in a finite time. This is known as the Big Rip. As mentioned above the equation of state parameter with super negative values leads to Big Rip which can be avoided in a particular class of models. Phantom states are unphysical states since they have they have a negative kinetic energy and therefore these states roll up the potential. This process will go on to infinity, unless the state reaches a maximum of its

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potential. If the phantom potential has no maximum, the phantom energy is rolling up to infinity which leads to a curvature of the universe growing to infinity in a finite time. The phantom fluid will rip everything apart. The phantom energy becomes the dominant force that will overrule all known fundamental forces. This scenario is called the Big Rip. This Big Rip singularity can be avoided if the potential has a maximum.

The physical properties of phantom energy are rather weird as they include violation of the dominant energy condition and increasing energy density with the universe expands in a super accelerated fashion, so as the big rip and the possibility for a big trip. The energy density of phantom energy increases with time. The phantom energy density becomes infinite in finite time, which will rip apart the universe. By assuming a particular relation between the time derivative of the phantom field and the Hubble function, an exact solution of the model is constructed. The scale factor 'a' and the Hubble parameter 'H' reach infinity in finite time and the universe ends in an explosive 'big rip', a final singularity in which the universe is destroyed in a finite proper time by excessive expansion. Comparing such model with the current cosmological observations shows that its predictions are consistent with all astrophysical observations.

The exact solution for a phantom scalar field with an exponential potential

The dynamics of the cosmological evolution is characterized by the Hubble variable

$$h = \frac{\dot{a}}{a}$$

Where a is the cosmological radius of the universe.

Using the exponential scalar potential energy density:

$$V(\phi) = V_0 e^{\lambda\phi}$$

The parameter λ will ultimately determine the resulting cosmology of the system. It will determine whether or not the system is acceleration and what species will be dominant at a particular time.

Power law is designed to self- consistently solve for the evolution of scalar fields and the scale factor in an expanding universe. In some cases, however, it may wish to solve for the behavior of a set of fields in a universe dominated by other forms of energy, e.g. pure matter or radiation. Exact solution of the field equations are obtained by the assumption of power-law form of the scale factor.

In standard cosmology, these results, when combined with the latest CMB data and clustering estimates, are used to make out a case for a universe in which accelerated expansion is fueled by a self-interacting, unclustered fluid, with high negative pressure, collectively known as dark energy, the simplest and the most favoured candidate being the cosmological constant Λ. Such a discrepancy between theoretical expectations and empirical observations constitute a fundamental problem at interface of astrophysics, cosmology and particle physics.

Also using a power law for the scale factor a, as well as the following ansatz for the scalar field:

$$a = a_0 t^k$$

Suppose that the scalar field has a time dependence,

$$\phi(t) = \phi_0 \ln(t) + \phi_1$$

The Hubble parameter can be written, thus

$$H = \frac{\dot{a}}{a} = \frac{k}{t}$$

Also, rewriting the scalar potential:

$$V = V_0 t^{\lambda\phi_0} e^{\lambda\phi_1}$$

Thus, computing the elements of the Klein-Gordon equations:

$$\dot{\phi} = \frac{\phi_0}{t}$$

$$\ddot{\phi} = -\frac{\phi_0}{t^2}$$

Klein-Gordon equation acquires the form,

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \lambda V_0 e^{\lambda\phi} = 0 \qquad (1)$$

$$-\frac{\phi_0}{t^2} = -3\left(\frac{k}{t}\right)\frac{\phi_0}{t} - V_0\lambda t^{\lambda\phi_0}e^{\lambda\phi_1}$$

But, $\phi_0 = -\frac{2}{\lambda}$

$$-\frac{2}{\lambda}\frac{(3k-1)}{t^2} + \frac{\lambda V_0 e^{\lambda\phi_1}}{t^{-\lambda(-\frac{2}{\lambda})}} = 0$$

$$\frac{6k}{\lambda^2} = -\frac{2}{\lambda^2} + V_0 e^{\lambda\phi_1} \qquad (2)$$

Considering the phantom scalar field with a negative kinetic term, we shall have the following Friedmann equation,

$$\frac{\dot{a}^2}{a^2} = -\frac{\dot{\phi}^2}{2} + V_0 e^{\lambda\phi} \qquad (3)$$

The Friedmann equation gives now

$$\frac{\dot{a}^2}{a^2} = -\frac{\dot{\phi}^2}{2} + V_0 e^{\lambda\phi}$$

$$\left(\frac{k}{t}\right)^2 = -\frac{{\phi_0}^2}{2t^2} + V_0 t^{\lambda\phi_0}e^{\lambda\phi_1}$$

$$k^2 = -\frac{2}{\lambda^2} + V_0 e^{\lambda\phi_1} \qquad (4)$$

Combining eqn (2) and eqn (4),

$$k^2 = \frac{6k}{\lambda^2} \qquad (5)$$

$$|\lambda| < 3\sqrt{2}$$

Thus, to solve the equation, we can have $k = \frac{1}{3}$, which is the law of expansion of the universe filled with stiff matter or massless scalar field, which in turn, means that $V_0 = 0$.

Substituting eqn. (5) into the relation eqn. (4),

$$(\frac{6}{\lambda^2})^2 = -\frac{2}{\lambda^2} + V_0 e^{\lambda\phi_1}$$

$$V_0 e^{\lambda\phi_1} = \frac{36}{\lambda^4} + \frac{2}{\lambda^2}$$

$$e^{\lambda\phi_1} = \left[\frac{2(18+\lambda^2)}{\lambda^4 V_0}\right]$$

$$\phi_1 = \frac{1}{\lambda}\ln[\frac{2(18+\lambda^2)}{\lambda^4 V_0}] \qquad (6)$$

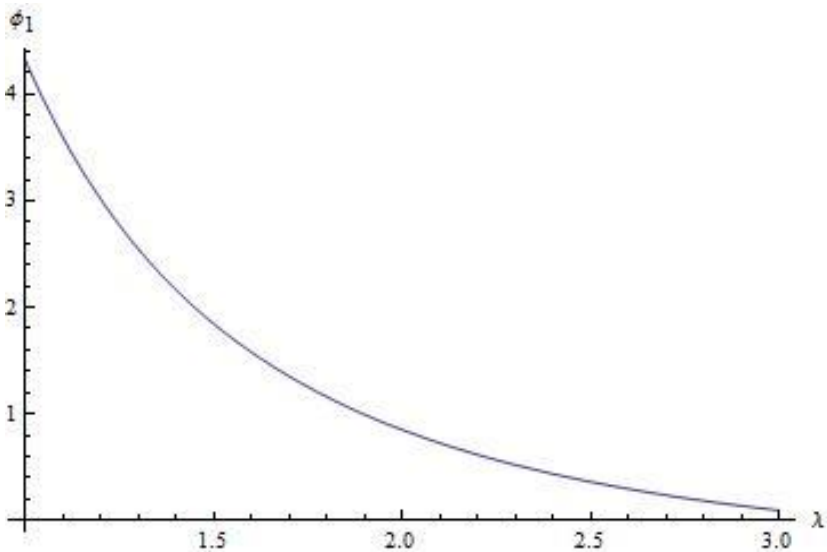

Figure 1: Variation of scalar field ϕ_1 with λ

The particular exact solution for the phantom case

We shall look for the solution of the phantom scalar field, which depends logarithmically on the cosmic time t, while the scale factor undergoes a power-law expansion (or contraction).

The explicit form of the exact particular solution,

$$\phi(t) = \phi_0 \ln t + \phi_1$$

If we would like to consider an expanding universe, then the solution will be,

$$\phi(t) = -\frac{2}{\lambda}\ln(-t) + \frac{1}{\lambda}ln\left[\frac{2(18+\lambda^2)}{\lambda^4 V_0}\right] \tag{7}$$

where t is running from $-\infty\ to\ 0$. The Hubble parameter is now

$$h(t) = \frac{\dot{a}}{a} = \frac{k}{-t}$$

$$= -\frac{6}{\lambda^2 t} \tag{8}$$

Thus, the formulae (7) and (8) describe a cosmological evolution which begins at $t = -\infty$ and ends at $t = 0$, encountering a Big Rip singularity. However, another particular solution describes a cosmological evolution which begins at $t = 0$ and ends at $t = \infty$, phantom field leads the universe to accelerate its expansion.

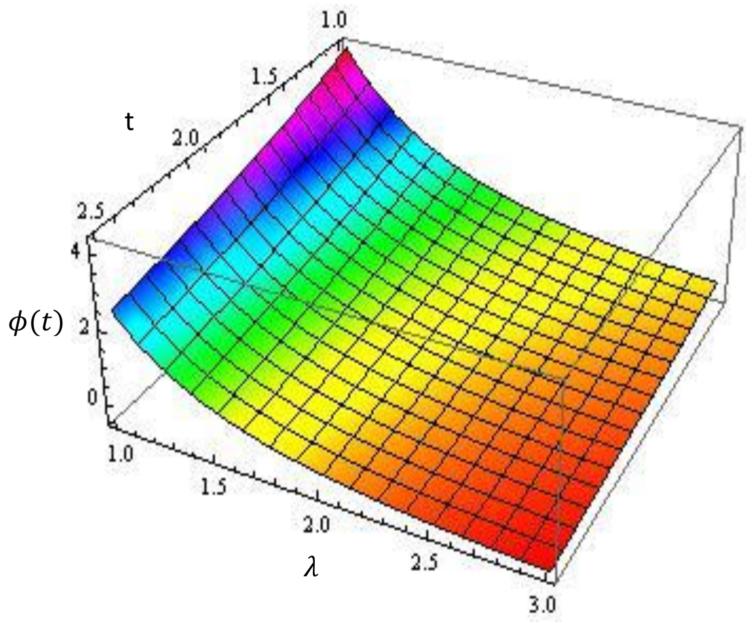

Figure 2: 3D variation of $\phi\ (t)$ with the time t and λ

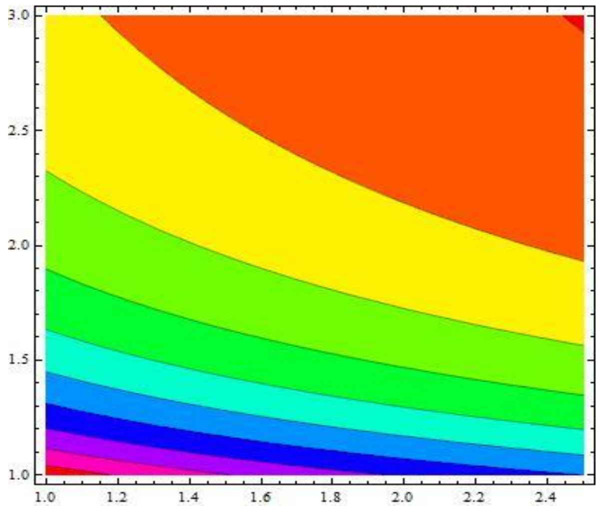

Figure 3: Contour profile of scalar field $\phi\ (t)$

The general exact solution for the phantom case

We shall introduce the now variables $v\ and\ u,$ such that

$$a^3 = e^{v+u} \tag{9}$$

$$\phi = A(v-u) \tag{10}$$

$$A = \frac{\sqrt{2}}{3} \tag{11}$$

where A is a coefficient to be defined.

The Friedmann equation is

$$\frac{\dot{a}^2}{a^2} = -\frac{\dot{\phi}^2}{2} + V_0 e^{\lambda\phi}$$

However, because of the negative sign of the kinetic term in the right-hand side of the Friedmann equation,

$$a^2 = (e^{v+u})^{\frac{2}{3}}, \quad \dot{a}^2 = \frac{1}{9}(\dot{v} + \dot{u})^2 (e^{v+u})^{2/3}, \ \dot{\phi}^2 = A^2(\dot{v} - \dot{u})^2$$

We get, $\dot{u}^2 + \dot{v}^2 = \frac{9}{2}V_0 e^{\lambda\phi}$ (12)

It is convenient now to introduce a complex variable

$$z \equiv \frac{1}{\sqrt{2}}(v + iu)\ ,\ \bar{z} \equiv \frac{1}{\sqrt{2}}(v - iu) \quad (13)$$

Now eqn. (12) looks like, $\dot{z}\,\dot{\bar{z}} = \frac{9}{4}V_0 e^{\lambda\phi}$ (14)

where "bar" stands for the complex conjugation.

However, $z' \equiv \frac{1}{\sqrt{2}}(v' + iv')\ ,\ \bar{z}' \equiv \frac{1}{\sqrt{2}}(v' - iu')$

We obtain, $z'\bar{z}' = 1$ (15)

Rewriting the Klein-Gordon equation,

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \lambda V_0 e^{\lambda\phi} = 0$$

and taking into account the above relation (15) we come to

$$z'' + \frac{\sqrt{2}(1-i)}{2}\left[z'^2\left(1 + \frac{\sqrt{2}\lambda i}{6}\right) - i - \frac{\sqrt{2}\lambda}{6}\right] = 0 \quad (16)$$

Introducing the function f such that

$$z' \equiv \frac{1}{\alpha}\frac{f'}{f} \quad (17)$$

where , $\alpha = \frac{\sqrt{2}(1-i)}{2}\ (1 + \frac{\sqrt{2}\lambda i}{6})$ (18)

From eqn.(16),

$$z'' + \left(1 + \frac{\sqrt{2}\lambda}{6}\right)z'^2 + \left(\frac{\sqrt{2}\lambda}{6} - 1\right) = 0$$

Introducing a new variable, $x = z'$

Now, the Klein-Gordon equation is

$$x' + \left(1 + \frac{\sqrt{2}\lambda}{6}\right)x^2 + \left(\frac{\sqrt{2}\lambda}{6} - 1\right) = 0 \quad (19)$$

Where, $x = \frac{1}{(1+\frac{\sqrt{2}\lambda}{6})}\frac{f'}{f}$

$$x' = \frac{1}{(1+\frac{\sqrt{2}\lambda}{6})}\left[\frac{ff''-f'^2}{f^2}\right]$$

Differentiating with respect to time and the value of x and x', eqn.(19) yields

$$f'' - \left(1 + \frac{\lambda^2}{18}\right) f = 0$$

The auxiliary function f satisfies the following equation,

$$f'' - \bar{\kappa}^2 f = 0 \qquad (20)$$

where , $\bar{k} \equiv \sqrt{1 + \frac{\lambda^2}{18}}$

The general solution of eqn. (20) is

$$f'' = \left(1 + \frac{\lambda^2}{18}\right) f$$

We have defined $f = Fe^{\bar{\kappa}\tau} + Ge^{-\bar{\kappa}\tau}$ (21)

(See Appendix)

Using eqn.(17), $z' = \frac{1}{(1+\frac{\sqrt{2}\lambda}{6})}\frac{f'}{f}$

$$\int dz = \frac{1}{(1+\frac{\sqrt{2}\lambda}{6})}\int \frac{df}{f}$$

$$z = \frac{1}{(1+\frac{\sqrt{2}\lambda}{6})} lnf + z_0$$

$$z = \frac{1}{(1+\frac{\sqrt{2}\lambda}{6})} \ln(Fe^{\bar{\kappa}\tau} + Ge^{-\bar{\kappa}\tau}) + z_0 \qquad (22)$$

Using the relation eqn.(22) we can analogously find

$$\bar{z} = \frac{1}{(1-\frac{\sqrt{2}\lambda}{6})} \ln(Fe^{\bar{\kappa}\tau} - Ge^{-\bar{\kappa}\tau}) + \bar{z}_0 \qquad (23)$$

$$\phi = A(z - \bar{z}),\ A = \frac{\sqrt{2}}{3}$$

Substituting eqn.(22) and eqn.(23),

$$\phi(\tau) = \frac{\sqrt{2}}{3}\left[\frac{1}{(1+\frac{\sqrt{2}\lambda}{6})}\ln(Fe^{\bar{\kappa}\tau} + Ge^{-\bar{\kappa}\tau}) + z_0 - \frac{1}{\left(1-\frac{\sqrt{2}\lambda}{6}\right)}\ln(Fe^{\bar{\kappa}\tau} - Ge^{-\bar{\kappa}\tau}) + \bar{z}_0\right]$$

$$\phi(\tau) = \frac{\sqrt{2}}{3}\left[(z_0 + \bar{z}_0) + \frac{1}{(1+\frac{\sqrt{2}\lambda}{6})}\ln(Fe^{\bar{\kappa}\tau} + Ge^{-\bar{\kappa}\tau}) - \frac{1}{\left(1-\frac{\sqrt{2}\lambda}{6}\right)}\ln(Fe^{\bar{\kappa}\tau} - Ge^{-\bar{\kappa}\tau})\right]$$

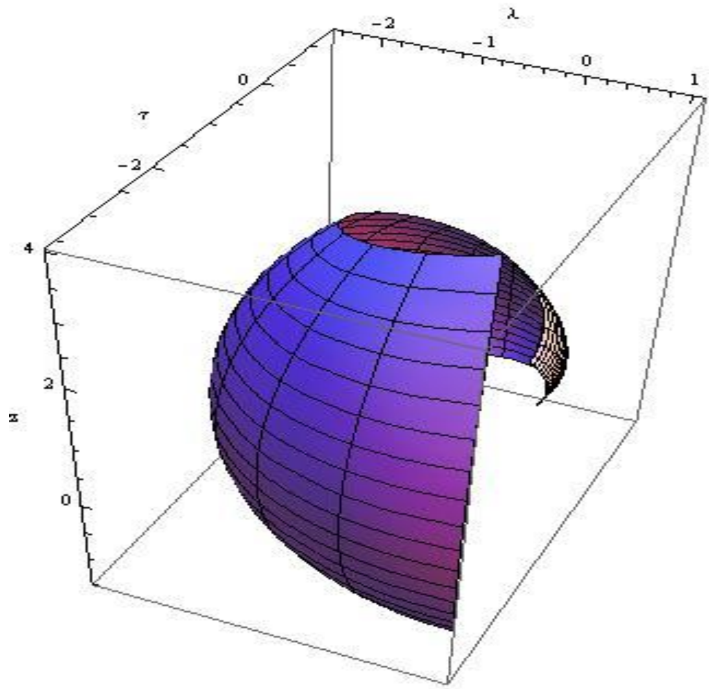

Figure 4: The spherical plot profile of field function z

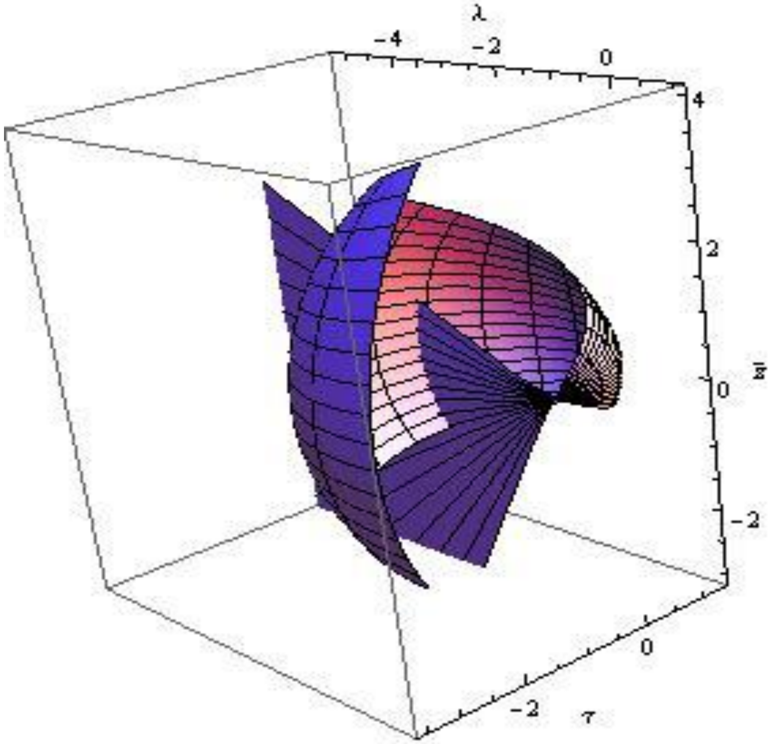

Figure 5: The spherical plot profile of field function $\bar{z}$

First, the case $G = 0$, $F \neq 0$ gives the first particular exact solution (eqn. 7) as it should be and in complete analogy with the hyperbolic case. We get,

$$\frac{\sqrt{2}}{3}(z_0 - \bar{z}_0) = -\frac{2\lambda}{9\bar{\kappa}^2}\ln|f|$$

The case $F = 0$, while $G \neq 0$gives the second particular exact solution, describing an infinite contraction of the universe, which begins at the anti-Big Rip singularity.

Now, we consider the case when both the constants $F\ and\ G$ are different from zero.We can choose one of these constants, say $F = 1\ while\ G = i$.

$$\phi(\tau) = -\frac{\sqrt{2}}{3}(z_0 - \bar{z}_0) + \frac{2\sqrt{2}}{6+\sqrt{2}\lambda}\ln(1+ie^{-2\bar{\kappa}\tau}) - \frac{2\sqrt{2}}{6-\sqrt{2}\lambda}\ln(1-ie^{-2\bar{\kappa}\tau})$$

$$\phi(\tau) = \frac{2\lambda}{9\bar{\kappa}^2}\ln|f| + \ln(1+ie^{-2\bar{\kappa}\tau})\frac{36\sqrt{2}}{3(18-\lambda^2)} + const$$

$$= \frac{2\lambda}{9\bar{\kappa}^2}\ln|f| - \ln(1+ie^{-2\bar{\kappa}\tau})\frac{2\sqrt{2}}{3\bar{\kappa}^2} + const$$

Now, we can find the expression for the scalar field

$$\phi(\tau) = \frac{2\lambda}{9\bar{\kappa}^2}\ln|f| - \frac{2\sqrt{2}}{3\bar{\kappa}^2}\arg f + const \qquad (24)$$

When $\tau \to -\infty$ the field $\phi \to +\infty$, while when $\tau \to +\infty$ the field $\phi \to +\infty$ again. Thus, the beginning and the end of the cosmological evolution are characterized by a positive infinite value of the scalar field and, hence by the positive infinite value of the potential V.

The complete cosmological evolution involves a finite period of the cosmic time t. This solution beginning from “anti-Big Rip” singularity and ending in the Big Rip singularity, passing through the point of minimal contraction was not considered in this paper devoted to the study of phantom solutions with exponential potentials.

In figure (7) we represent two particular exact solution and a typical example of general solution for the phantom case.

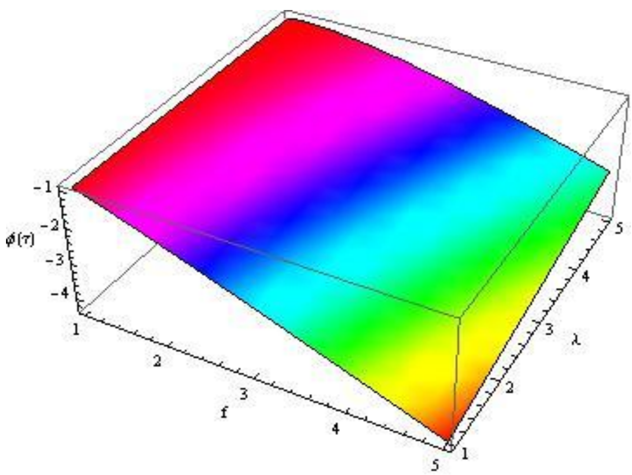

Figure 6: 3D variation of ϕ (τ) with f and λ

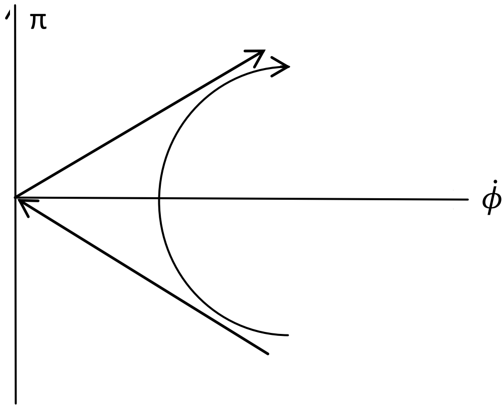

Figure 7: The phase space diagram $\dot{\phi}, \pi,$ where π stays for the velocity $\pi = \dot{\phi}$.Two straight line trajectories describe two particular solutions, corresponding to the expanding and contracting universe. The curve line represents a trajectory, belonging to the family of those evolving from the anti-Big Rip singularity to the Big Rip singularity, passing through the point of minimal contraction of the universe.

Results and Concluding Remarks

Attempts have been made to describe the general cosmological solution with an exponential in contrast to the "old" particular solution for a phantom scalar field and construct the general solution for piecewise exponential potential with cusps. One can say that these finite time evolutions represent the phantom counterpart of the well-known cosmological evolutions, which begin in the Big Bang singularity reach the point of maximal expansion and then have a stage of contraction culminating in the Big Crunch singularity. The contribution of phantom field leads the universe to reach the critical energy and to accelerate its expansion. The simplest explanation for the phantom dark energy is provided by a scalar field with a negative kinetic energy. The exact solution of the Friedmann and Klein Gordon equations are derived by assuming a particular relation between the time derivative of the phantom field and the Hubble function and then determine the evolution of the expansion scale factor and the potential from it. Our detailed analysis shows that in the model under consideration the phantom field can be an excellent candidate for dark energy.

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References

Andrianov. A.A, Cannata.F, Kamenshchik.A.Y and Regoli.D, (2010), 'Phantom Cosmology Based On PT Symmetry', Int. J. Mod. Phys. D 19 97.

Cai. R.G and Wang.A, (2005), 'Cosmology With Interaction Between Phantom Dark Energy and Dark Matter and the coincidence problem', JCAP 0503 002 [hep-th/0411025].

Caldwell. R.R, Kamionkowski.M and Weinberg.N.N, (2003), 'Phantom Energy and Cosmic Doomsday', Phys. Rev. let. 91071301 [astro-ph/0302506].

Cannata. F. and Kamenshchik.A.Yu, (2007), 'Networks of Cosmological Histories, Crossing of the Phantom Divide Line and Potentials with Cusps', Int. J. Mod. Phys. D 16 1683 [gr-qc/0603129].

Appendix

The exact solution for a phantom scalar field with an exponential potential

The Friedmann equation is

$$\frac{\dot{a}^2}{a^2} = -\frac{\dot{\phi}^2}{2} + V_0 e^{\lambda\phi} \tag{3}$$

From eqn.(9), $a^3 = e^{v+u}$

$$a^2 = (e^{v+u})^{\frac{2}{3}}$$

$$\dot{a}^2 = \frac{1}{9}(\dot{v} + \dot{u})^2 (e^{v+u})^{2/3}$$

$$\frac{\dot{a}^2}{a^2} = \frac{1}{9}(\dot{v} + \dot{u})^2$$

$$\dot{\phi}^2 = A^2(\dot{v} - \dot{u})^2$$

Now, eqn.(3) becomes

$$\therefore\ \frac{1}{9}(\dot{u}^2 + \dot{v}^2 + 2\dot{u}\dot{v}) = -\frac{A^2}{2}(\dot{u}^2 + \dot{v}^2 - 2\dot{u}\dot{v}) + V_0 e^{\lambda\phi}$$

The Friedmann equation (3) has now the form

$$\frac{1}{9}(\dot{u}^2 + \dot{v}^2 + 2\dot{u}\dot{v}) = -\frac{1}{9}(\dot{u}^2 + \dot{v}^2 - 2\dot{u}\dot{v}) + V_0 e^{\lambda\phi}$$

$$\dot{u}^2 + \dot{v}^2 = \frac{9}{2} V_0 e^{\lambda\phi} \tag{12}$$

It is convenient now to introduce a complex variable

$$z \equiv \frac{1}{\sqrt{2}}(v + iu), \bar{z} \equiv \frac{1}{\sqrt{2}}(v - iu) \tag{13}$$

From eqn.(12),

$$\frac{1}{2}(\dot{u}^2 + \dot{v}^2) = \frac{9}{4} V_0 e^{\lambda\phi}$$

$$\frac{1}{\sqrt{2}}(\dot{v} + i\dot{u})\frac{1}{\sqrt{2}}(\dot{v} - i\dot{u}) = \frac{9}{4} V_0 e^{\lambda\phi}$$

$$\dot{z} = \frac{1}{\sqrt{2}}(\dot{v} + i\dot{u})$$

$$\dot{\bar{z}} = \frac{1}{\sqrt{2}}(\dot{v} - i\dot{u})$$

$$\dot{z}\dot{\bar{z}} = \frac{1}{2}(\dot{v}^2 + \dot{u}^2)$$

$$= \frac{1}{2} \times \frac{9}{2} V_0 e^{\lambda\phi}$$

Now eqn. (12) looks like, $\dot{z}\,\dot{\bar{z}} = \frac{9}{4} V_0 e^{\lambda\phi}$ (14)

Another complex variable, $z' = \frac{1}{\sqrt{2}}(v' + iu')$

$$\bar{z}' = \frac{1}{\sqrt{2}}(v' - iu')$$

We obtain, $z'\bar{z}' = 1$ (15)

Introducing the function f such that

$$z' \equiv \frac{1}{\alpha}\frac{f'}{f} \qquad (17)$$

where , $\alpha = \frac{\sqrt{2}(1-i)}{2}\left(1 + \frac{\sqrt{2}\lambda i}{6}\right)$ (18)

Now, substituting eqn.(15) and eqn.(18) into eqn.(16).We get,

$$z'' + \left(1 + \frac{\sqrt{2}\lambda}{6}\right) z'^2 + \left(\frac{\sqrt{2}\lambda}{6} - 1\right) = 0$$

Introducing a new variable, $x = z'$

$$\therefore x' + \left(1 + \frac{\sqrt{2}\lambda}{6}\right) x^2 + \left(\frac{\sqrt{2}\lambda}{6} - 1\right) = 0 \qquad (19)$$

This Klein-Gordon equation can be satisfied the Raccati form;

$$x = \frac{1}{(1+\frac{\sqrt{2}\lambda}{6})}\frac{f'}{f}$$

$$x' = \frac{1}{(1+\frac{\sqrt{2}\lambda}{6})}\left[\frac{ff''-f'^2}{f^2}\right]$$

Substituting into eqn.(19),

$$\frac{1}{(1+\frac{\sqrt{2}\lambda}{6})}\left[\frac{ff''-f'^2}{f^2}\right] + \frac{(1+\frac{\sqrt{2}\lambda}{6})}{(1+\frac{\sqrt{2}\lambda}{6})^2}\frac{f'^2}{f^2} + \left(\frac{\sqrt{2}\lambda}{6} - 1\right) = 0$$

$$\frac{1}{(1+\frac{\sqrt{2}\lambda}{6})}\frac{f''}{f}-\frac{1}{\left(1+\frac{\sqrt{2}\lambda}{6}\right)}\frac{f'^2}{f^2}+\frac{1}{\left(1+\frac{\sqrt{2}\lambda}{6}\right)}\frac{f'^2}{f^2}+(\frac{\sqrt{2}\lambda}{6}\text{-}1)=0 \quad (\times\left(1+\frac{\sqrt{2}\lambda}{6}\right)f)$$

$$f''+\left(\frac{\sqrt{2}\lambda}{6}-1\right)\left(1+\frac{\sqrt{2}\lambda}{6}\right)f=0$$

$$f''+\left(\frac{\sqrt{2}\lambda}{6}-\frac{2\lambda^2}{36}-1-\frac{\sqrt{2}\lambda}{6}\right)f=0$$

$$f''-\left(1+\frac{\lambda^2}{18}\right)f=0$$

$$f''-\bar{\kappa}^2 f=0$$

$$f''=\left(1+\frac{\lambda^2}{18}\right)f$$

$$\frac{d^2f}{d\tau^2}=\bar{\kappa}^2 f$$

$$\int\frac{df}{d\tau}=\pm\int\bar{\kappa}\,f$$

$$\ln f=\pm\,\bar{\kappa}\tau$$

$$f=\pm\,e^{\bar{\kappa}\tau}$$

$$f=Fe^{\bar{\kappa}\tau}+Ge^{-\bar{\kappa}\tau} \qquad (21)$$